1.2
Notes

DEFINITION OF LIMIT

OBJECTIVE: 1) Prove the limit of a function exists by use of the precise, formal definition of a limit.

THE PRECISE DEFINITION OF A LIMIT: (EPSILON–DELTA)

We say $\lim_{x \to a} f(x) = L$ **iff**: for every $\varepsilon > 0$, there is a $\delta > 0$ such that if

$|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

This definition can be difficult to understand. Do not be concerned if it does not make sense to you right now. The structure of this lesson is designed to help you understand the definition and be able to apply it in some cases. We will start with a graphical interpretation of the precise definition. Then we will explore the algebraic manipulations involved in applying the precise definition to specific examples.

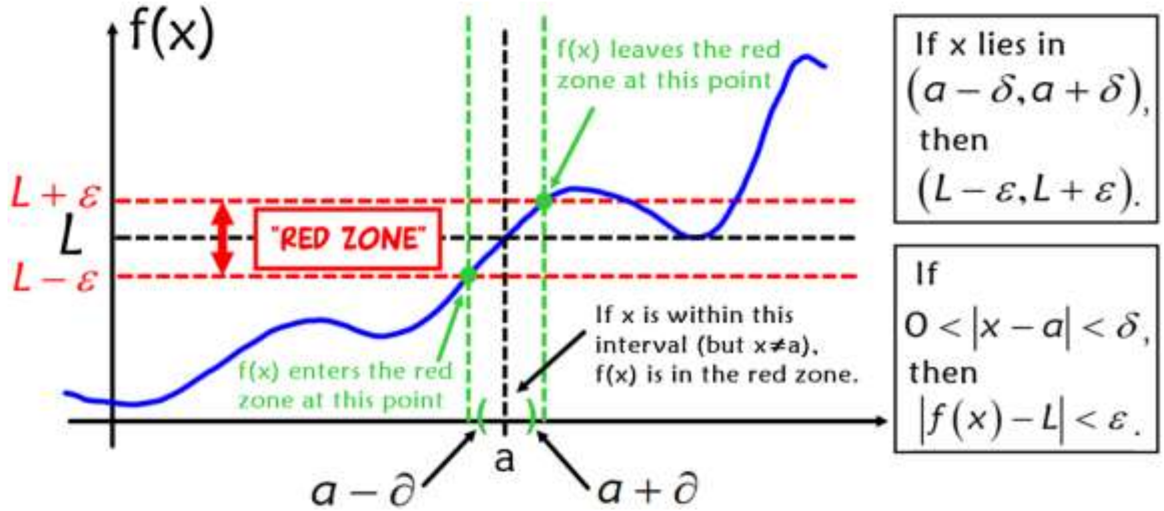
First, consider what this means: $|x - 3| < .01$

$x - 3 < .01$
$x < 3.01$

$-(x-3) < .01$
$-x + 3 < .01$
$-x < -2.99$
$x > 2.99$

$\boxed{2.99 < x < 3.01}$

GRAPHICAL INTERPRETATION



ALTERNATIVE DEFINITION OF A LIMIT: (EPSILON–DELTA)

We say $\lim_{x \to a} f(x) = L$ **iff**: for every $\varepsilon > 0$, there is a $\delta > 0$ such that if x is in the open interval $(a - \delta, a + \delta)$ and $x \neq a$ then $f(x)$ is in the open interval $(L - \varepsilon, L + \varepsilon)$.

The formal definition says we can make f(x) as "close as we like" to L by making x "close enough (but not equal to)" to a. In other words, for each and every epsilon we choose, we must be able to find a delta.

APPLYING THE DEFINITION: FINDING THE LARGEST DELTA FOR A GIVEN EPSILON

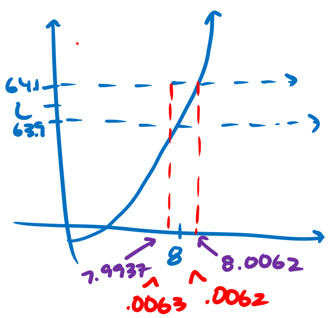
Given: $\lim_{x\to 8} x^2 - 2 = 62$. If $\varepsilon = .1$ find the largest value of δ. Round your answer four decimals.

If $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

$|x^2 - 2 - 62| < .1$

$|x^2 - 64| < .1$

$63.9 < x^2 < 64.1 \Rightarrow 7.9937 < x < 8.0062$



$\boxed{\delta = .0062}$

PROVING LIMITS USING THE DEFINITION

Example: Use the precise definition of a limit to prove the following: $\lim_{x\to 2}(3x+5) = 11$

1) **SCRATCH WORK**: Find δ by letting ε be an arbitrary number. Use the inequality $|f(x) - L| < \varepsilon$ to work backwards to a statement of the form $|x-a| < \delta$.

$|3x + 5 - 11| < \varepsilon$

$|3x - 6| < \varepsilon$

$3|x-2| < \varepsilon$

$|x-2| < \frac{\varepsilon}{3}$

$\boxed{\delta \le \frac{\varepsilon}{3}}$

2) **ARGUMENT:** Write the proof. For every $\varepsilon > 0$, assume $0 < |x-a| < \delta$ and use the work in Step 1 to prove that $|f(x) - L| < \varepsilon$. (Work forwards)

If $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

$0 < |x-2| < \frac{\varepsilon}{3}$

$0 < 3|x-2| < \varepsilon$

$0 < |3x-6| < \varepsilon$

$0 < |(3x+5) - 11| < \varepsilon$ QED

PROVING A LIMIT DOES NOT EXIST

Use the definition of a limit to prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. (INDIRECT PROOF)

Suppose that $\lim_{x \to 0} \frac{|x|}{x} = 1$. Then we should be able to find an open interval $(0-\delta, 0+\delta) \Rightarrow (-\delta, \delta)$ where if $-\delta < x < \delta$, then $\left(x, \frac{|x|}{x}\right)$ lies between $(L-\varepsilon, L+\varepsilon)$. However this is not true for $-\delta < x < 0$ and therefore our assumption is false. Therefore the $\lim_{x \to 0} \frac{|x|}{x}$ D.N.E.