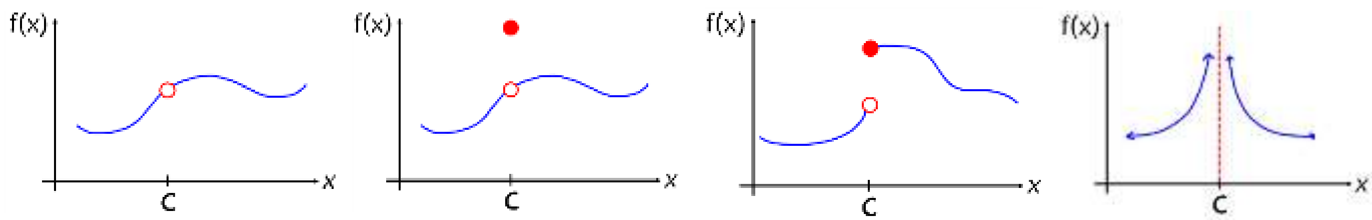
CONTINUOUS FUNCTIONS

OBJECTIVES: 1) Determine if a function is continuous at some value of x.
2) Use the Intermediate Value Theorem to prove the existence of a zero of a function.

In this section we use limits to define continuous functions and examine their behavior. First, here's a look at functions that are NOT continuous at some value c.



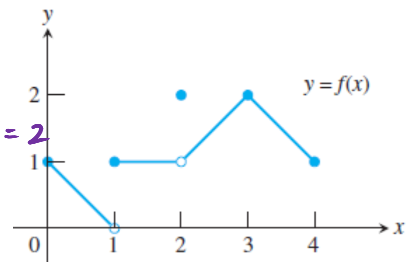
$\lim_{x\to c} f(x) \neq f(c)$ $\quad$ $\lim_{x\to c} f(x) \neq f(c)$ $\quad$ $\lim_{x\to c} f(x) \to$ D.N.E $\quad$ $\lim_{x\to c} f(x) \to$ D.N.E

DEFINITION: A function $f(x)$ is continuous at $x = c$ if and only if $\lim_{x\to c} f(x) = f(c)$.

Is $f(x)$ continuous at c? You must show these steps when asked to justify/prove/show continuity at a point!

	x = 1	x = 2	x = 3
1) Evaluate $f(c)$	$f(1) = 1$	$f(2) = 2$	$f(3) = 2$
2) Evaluate $\lim_{x\to c} f(x)$	$\lim_{x\to 1} f(x) =$ DNE	$\lim_{x\to 2} f(x) = 1$	$\lim_{x\to 3} f(x) = 2$

3) Determine if $\lim_{x\to c} f(x) = f(c)$.
- ✓ Does step 1 equal step 2?
- ✓ If yes, then $f(x)$ is continuous at $x = c$.



FACT: 1) Polynomial functions are continuous at every real number c.
2) Rational functions are continuous at every real number in their domain.
*Note: Trigonometric, logarithmic, and exponential functions are also continuous at every real number in their domain.

Let a function $f(x)$ be defined on a closed interval $[a,b]$. We say that $f(x)$ is "**continuous on** $[a,b]$" if it is continuous on (a,b), and if, in addition, $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

Ex 1) Show that f is continuous at c.

$f(x) = \dfrac{\sqrt[3]{x}}{2x+1}$ $\quad$ c = 8

$f(8) = \dfrac{\sqrt[3]{8}}{2(8)+1} = \boxed{\dfrac{2}{17}}$ $\quad$ $\lim_{x\to 8} \dfrac{\sqrt[3]{x}}{2x+1} = \boxed{\dfrac{2}{17}}$

$f(8) = \lim_{x\to 8} f(x)$ $\therefore$ f is continuous @ $x = 8$.

Ex 2) Find the values for which the function is continuous.

a) $g(x) = \dfrac{x^2 - 9}{2x+1}$

$(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

b) $h(x) = \sqrt{16 - x^2}$

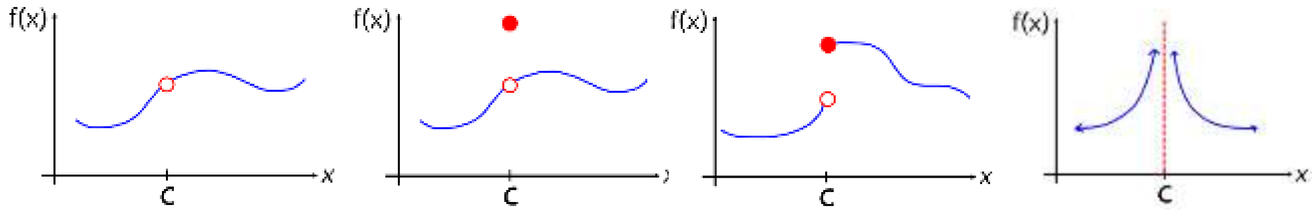
$16 - x^2 \geq 0$

$[-4, 4]$

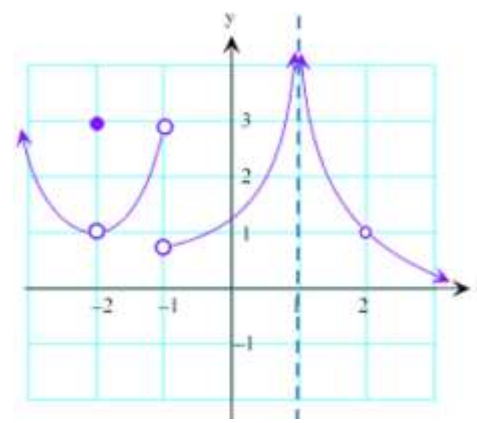
Now let's return to our graphs to discuss graphs of functions with discontinuities a bit more.

TYPES OF DISCONTINUITIES:

REMOVABLE | **JUMP** | **INFINITE**



3) Name the type of discontinuity at each value of x on the graph of this piecewise function.



a) $x = -2$ Removable Discontinuity

$f(-2) = 3$ $\lim_{x \to -2} f(x) = 1$

b) $x = -1$ Jump

c) $x = 1$ Infinite

d) $x = 2$ Removable

4) Determine where the function, $g(x) = \dfrac{x-5}{x^2 - 2x - 15}$, is not continuous and specify the type of discontinuity.

g(x) is discontinuous when the denominator is zero. $0 = x^2 - 2x - 15$

$0 = (x-5)(x+3)$

$x = 5, -3$

It is discontinuous at $x = 5$ & $x = -3$.

There is a hole at $x = 5$, so there is a removable disc. At $x = -3$, there is a vertical asymptote. At $x = -3$ → infinite disc.

5) Determine the value of "*a*" that would make the function continuous.

a) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$

b) $h(x) = \begin{cases} ax + 1, & x \leq 3 \\ ax^2 - 1, & x > 3 \end{cases}$

$\dfrac{\sin x}{x}$ is discontinuous when $x = 0$, ∴ to make f(x) continuous at $x = 0$, we need to define a, to be $a = 0$. Now $f(0) = 0$ and $\lim_{x \to 0} f(x) = 0$.

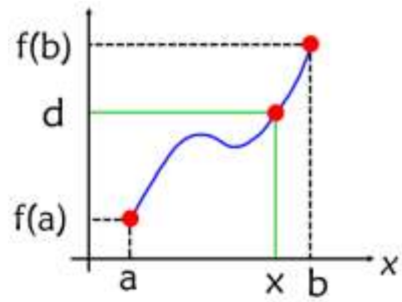
$ax + 1 = ax^2 - 1$

$3a + 1 = 9a - 1$

$6a = 2$

$\boxed{a = \frac{1}{3}}$

INTERMEDIATE VALUE THEOREM: If $f(x)$ is continuous on the closed interval $[a,b]$ and d is a number such that $f(a) < d < f(b)$, then there exists at least one number x, such that $a < x < b$, such that $f(x) = d$.



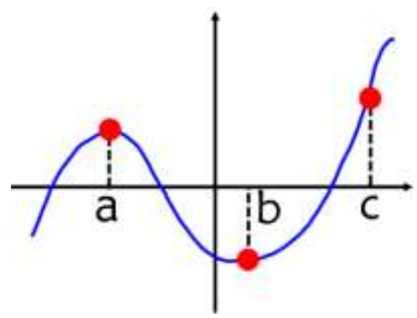
SIMPLY PUT: If *f(x)* is continuous from *a* to *b*, then it takes on every value between *f(a)* and *f(b)*.

6) Show that for $g(x) = x^3 - 5x^2 + 8x - 9$ there is some x, $0 < x < 5$, such that $g(x) = 27$.

$g(0) = -9$
$g(5) = 31$

Since $g(x)$ is a polynomial, it's continuous for all reals. Since $g(0) < 27 < g(5)$, then by the Intermediate Value theorem, there exists an x, where $0 < x < 5$, such that $g(x) = 27$.

IMPORTANT RESULT: If *f(x)* is continuous on [a, b], and if *f(a)* and *f(b)* have **opposite signs**, then there is at least one solution of the equation *f(x) = 0* in the interval (a, b).



7) Prove that the equation $f(x) = x^3 - 5x^2 + 8x - 9$ has at least one solution between 3 and 4.

$f(3) = -3$
$f(4) = 7$

The polynomial $f(x)$ is continuous for all reals. Since $f(3) < 0 < f(4)$, then by the Intermediate Value Theorem, there is an x, such that $3 < x < 4$, such that $f(x) = 0$.