10.4
Notes
(Part 2)

VECTORS: AN ALGEBRAIC APPROACH

OBJECTIVES: 1) Find the dot product of two vectors.
2) Find the angle between two vectors.

THE DOT PRODUCT (ALGEBRAIC)

Given two vectors $\boldsymbol{u} = \langle a_1, b_1 \rangle$ and $\boldsymbol{v} = \langle a_2, b_2 \rangle$, we define the **dot product** $\boldsymbol{u}\cdot\boldsymbol{v}$ as

$$\boldsymbol{u}\cdot\boldsymbol{v} = a_1 \cdot a_2 + b_1 \cdot b_2$$

Note: The dot product will yield a scalar (real number), which is why it's also referred to as a scalar product.

1) Find the dot product $\mathbf{s}\cdot\mathbf{t}$, if $\mathbf{s} = \langle -3, 4 \rangle$ and $\mathbf{t} = \langle 2, -5 \rangle$.

$s\cdot t = -3\cdot 2 + 4\cdot -5$

$= -6 + -20$

$= \boxed{-26}$

2) If $\mathbf{w} = 2\mathbf{i} + 2\mathbf{j}$ and $\mathbf{m} = 3\mathbf{i} - 3\mathbf{j}$, find the dot product.

$w\cdot m = 2(3) + 2(-3)$

$= 6 + -6$

$\boxed{0}$

PROPERTIES OF THE DOT PRODUCT

1) $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u}$

2) $\boldsymbol{u}\cdot(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{w}$

3) $\boldsymbol{u}\cdot\boldsymbol{u} = (|\boldsymbol{u}|)^2$

4) $a(\boldsymbol{u}\cdot\boldsymbol{v}) = a\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{u}\cdot a\boldsymbol{v}$

3) The magnitude of a vector is 7. What is $\mathbf{v}\cdot\mathbf{v}$?

$|v| = 7$

$v\cdot v = |v|^2 = 7^2 = 49$

$\boxed{v\cdot v = 49}$

4) Suppose $\mathbf{v}\cdot\mathbf{v} = 3$. What is the magnitude of $\mathbf{v}$?

$v\cdot v = 3 = |v|^2$

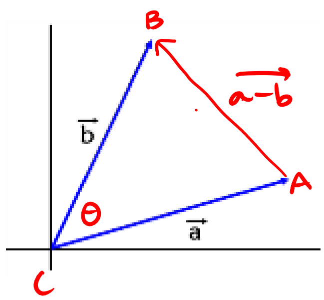
$\boxed{|v| = \sqrt{3}}$

THE DOT PRODUCT (GEOMETRIC)

Given two vectors $\boldsymbol{u} = \langle a_1, b_1 \rangle$ and $\boldsymbol{v} = \langle a_2, b_2 \rangle$, we define the **dot product** $\boldsymbol{u}\cdot\boldsymbol{v}$ as

$$\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}|\cdot|\boldsymbol{v}|\cos\theta$$

Proof: First, suppose that θ is the angle between two non-zero vectors as shown in the diagram below.



Using the law of cosines:

$$|a-b|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$$

$$|a-b|^2 = (a-b)\cdot(a-b) = a\cdot a - 2a\cdot b + b\cdot b$$
$$= |a|^2 - 2a\cdot b + |b|^2$$

$$\cancel{|a|^2} - 2a\cdot b + \cancel{|b|^2} = \cancel{|a|^2} + \cancel{|b|^2} - 2|a||b|\cos\theta$$

$$-2a\cdot b = -2|a||b|\cos\theta$$

$$a\cdot b = |a||b|\cos\theta$$

5) Given $|\mathbf{u}| = 8$ and $|\mathbf{v}| = 5$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{3\pi}{4}$, find $\mathbf{u}\cdot\mathbf{v}$.

$$u\cdot v = |u||v|\cos\left(\frac{3\pi}{4}\right)$$
$$= 8\cdot 5\left(-\frac{\sqrt{2}}{2}\right) = \boxed{-20\sqrt{2}}$$

FINDING THE ANGLE BETWEEN TWO VECTORS:

$$a\cdot b = |a||b|\cos\theta$$

$$\cos\theta = \frac{a\cdot b}{|a||b|} \qquad \theta = \cos^{-1}\left(\frac{a\cdot b}{|a||b|}\right)$$

6) Find the angle between $\mathbf{s} = \langle 3,0\rangle$ and $\mathbf{t} = \langle 1,6\rangle$.

$$s\cdot t = 3(1) + 0(6) = 3$$

$$|s| = \sqrt{3^2+0^2} = 3 \qquad |t| = \sqrt{1^2+6^2} = \sqrt{37}$$

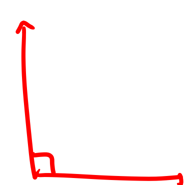
$$\theta = \cos^{-1}\left(\frac{3}{3\cdot\sqrt{37}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{37}}\right) \approx 80.5^\circ$$

ORTHOGONAL VECTORS: Two vectors are orthogonal (perpendicular) if $\mathbf{u}\cdot\mathbf{v} = 0$
*Perpendicular implies an intersection. Two vectors can be orthogonal to one another, they do not need to intersect.

***WHY?** If $\mathbf{u}\cdot\mathbf{v} = 0$, then:

$$\cos\theta = \frac{u\cdot v}{|u||v|} = \frac{0}{|u||v|} = 0$$

$$\cos\theta = 0$$
$$\theta = \cos^{-1}(0)$$
$$\theta = 90^\circ$$



7) Determine if the vectors are orthogonal.

a) $\mathbf{a} = \langle -2,5\rangle$ $\mathbf{b} = \langle 15,6\rangle$

$$a\cdot b = -2\cdot 15 + 5\cdot 6$$
$$= -30 + 30$$
$$= 0 \qquad \boxed{\text{YES!}}$$

b) $\mathbf{c} = 5\mathbf{i} - 2\mathbf{j}$ $\mathbf{d} = 3\mathbf{i} + 4\mathbf{j}$

$$c\cdot d = 5(3) + (-2)(4)$$
$$= 15 - 8$$
$$= 7 \qquad \boxed{\text{NO.}}$$