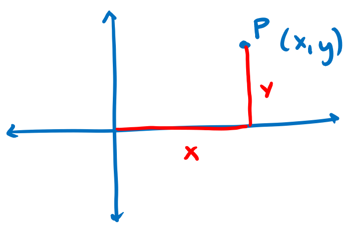
INTRODUCTION TO POLAR COORDINATES

OBJECTIVES: 1) Graph in polar coordinate form.
2) Change from polar to rectangular coordinates.

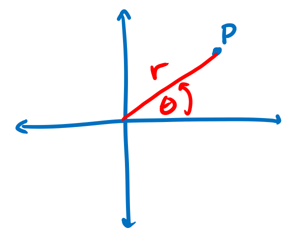
CARTESIAN VS. POLAR COORDINATES

Coordinate systems are used to describe the location of a point in space.

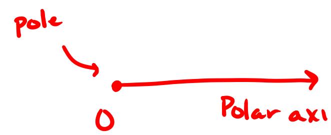
The Cartesian system describes how we should move from the origin both horizontally and vertically using coordinates (x,y).



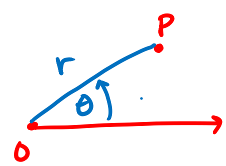
The polar system describes the distance straight from the origin to a point and determines the angle this segment makes with the positive x-axis.

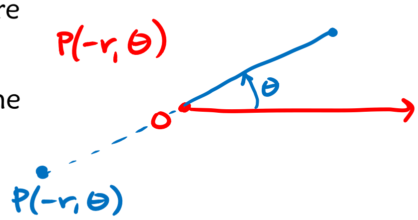


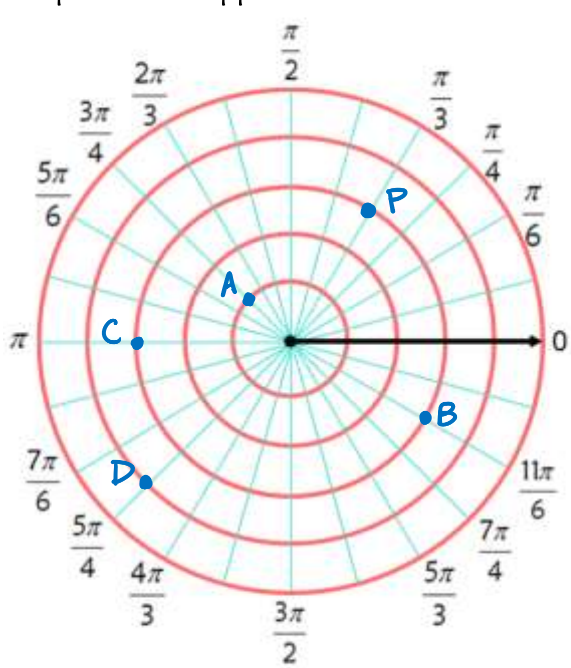
In the polar system, we begin with a fixed point O in the plane called the **pole** (or **origin**) and draw from O a ray called the **polar axis**. Then each point P can be assigned polar coordinates $P(r,\theta)$ where



- r is the distance from O to P
- θ is the angle between the polar axis and the segment $\overline{OP}$
- If θ is positive, we measure counter-clockwise from polar axis. If θ is negative, we measure in clockwise direction.
- If r is negative, then $P(r,\theta)$ is r units from the pole in the opposite direction of θ.







- Plot the following points:

$A\left(1,\frac{3\pi}{4}\right)$ $B\left(3,-\frac{\pi}{6}\right)$ $C(3,3\pi)$ $D\left(-4,\frac{\pi}{4}\right)$

- $P(r,\theta)$ can be represented by:

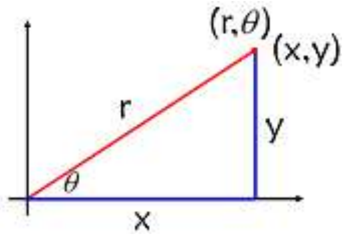
$P(r,\theta+2\pi n)$ or $P(-r,\theta+(2n+1)\pi)$

- If $P\left(2,\frac{\pi}{3}\right)$, list four other polar coordinates for P.

r > 0: $\left(2,\frac{7\pi}{3}\right)\left(2,-\frac{5\pi}{3}\right)$

r < 0: $\left(-2,\frac{4\pi}{3}\right)\left(-2,-\frac{2\pi}{3}\right)$

CONVERTING POLAR/RECTANGULAR COORDINATES



From the diagram, we arrive at the following relationships using Pythagorean Theorem and right-triangle trig:

$x^2+y^2=r^2$

$\cos\theta=\frac{x}{r}$ $\quad\sin\theta=\frac{y}{r}$ $\quad\tan\theta=\frac{y}{x}$

$r\cos\theta=x$ $\quad r\sin\theta=y$

1) Convert each rectangular point to polar coordinates:

a) $(-1, 1)$

$x^2+y^2=r^2$

$r=\pm\sqrt{2}$

$\tan\theta=\frac{y}{x}$ $\quad\tan\theta=-1$

$\theta=\frac{3\pi}{4}$ or $\frac{-\pi}{4}$

$(-1,1)$ lies in quad 2!

$\left(\sqrt{2},\frac{3\pi}{4}\right)$ or $\left(-\sqrt{2},-\frac{\pi}{4}\right)$

b) $(0, 2)$

$x^2+y^2=r^2$

$r=\pm 2$

$\tan\theta=\frac{2}{0}$ ← undefined!

$\theta=\frac{\pi}{2}$ or $-\frac{\pi}{2}$

$\left(2,\frac{\pi}{2}\right)$ or $\left(-2,-\frac{\pi}{2}\right)$

2) Given the polar coordinates, find the rectangular coordinates

a) $\left(5,\frac{2\pi}{3}\right)$

$x=r\cos\theta$ $\quad y=r\sin\theta$

$x=5\cos\left(\frac{2\pi}{3}\right)$ $\quad y=5\sin\left(\frac{2\pi}{3}\right)$

$x=5\left(-\frac{1}{2}\right)$ $\quad y=5\left(\frac{\sqrt{3}}{2}\right)$

$x=-\frac{5}{2}$ $\quad y=\frac{5\sqrt{3}}{2}$

$\left(-\frac{5}{2},\frac{5\sqrt{3}}{2}\right)$

b) $\left(\sqrt{3},\frac{\pi}{6}\right)$

$x=\sqrt{3}\cos\left(\frac{\pi}{6}\right)$ $\quad y=\sqrt{3}\sin\left(\frac{\pi}{6}\right)$

$x=\sqrt{3}\cdot\frac{\sqrt{3}}{2}$ $\quad y=\sqrt{3}\cdot\frac{1}{2}$

$x=\frac{3}{2}$ $\quad y=\frac{\sqrt{3}}{2}$

$\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right)$

CONVERTING A RECTANGULAR EQUATION TO A POLAR FORM

3) Convert the rectangular equation to polar form: $x^2=4y$

Use direct conversion

$(r\cos\theta)^2=4r\sin\theta$

$r^2\cos^2\theta=4r\sin\theta$

$r=\frac{4r\sin\theta}{r\cos^2\theta}$

$r=\frac{4\tan\theta}{\cos\theta}$

$\boxed{r=4\tan\theta\sec\theta}$

4) Convert the rectangular equation to polar form: $4x+7y-2=0$

$4r\cos\theta+7r\sin\theta-2=0$

$r(4\cos\theta+7\sin\theta)=2$

$\boxed{r=\frac{2}{4\sin\theta+7\sin\theta}}$

5) Convert the rectangular equation to polar form: $x^2 + y^2 - 8y = 0$

notice patterns: $x^2+y^2=r^2$

$r^2 - 8r\sin\theta = 0$

$r(r - 8\sin\theta) = 0$

$r - 8\sin\theta = \frac{0}{r}$

$\boxed{r = 8\sin\theta}$

CONVERTING A POLAR EQUATION TO RECTANGULAR FORM

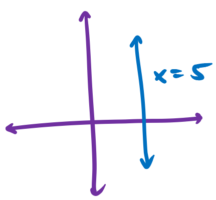
Convert the polar equation to rectangular form. If possible, determine the graph of the equation from its rectangular form.

6) $r = 5\sec\theta$

$r = \frac{5}{\cos\theta}$

$r\cos\theta = 5$

$x = 5$



7) $r = 2\sin\theta$

$r = 2\sin\theta$

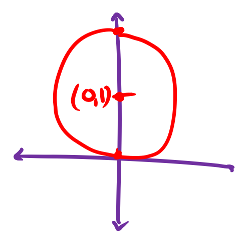
$r^2 = 2r\sin\theta$

$x^2+y^2 = 2y$

$x^2+y^2-2y = 0$

$x^2+y^2-2y+1 = 1$

$x^2+(y-1)^2 = 1$



Circle: center at $(0,1)$
radius 1

8) $r = 2 + 2\cos\theta$

$r^2 = 2r + 2r\cos\theta$

$x^2+y^2 = 2r + 2x$

$(x^2+y^2-2x)^2 = (2r)^2$

$(x^2+y^2-2x)^2 = 4r^2$

$(x^2+y^2-2x)^2 = 4(x^2+y^2)$

Yuck!

9) $r = \sin\left(\theta - \frac{\pi}{4}\right)$

$r = \sin\theta\cos\left(\frac{\pi}{4}\right) - \cos\theta\sin\left(\frac{\pi}{4}\right)$

$r = \frac{\sqrt{2}}{2}\sin\theta - \frac{\sqrt{2}}{2}\cos\theta$

$r^2 = \frac{\sqrt{2}}{2}r\sin\theta - \frac{\sqrt{2}}{2}r\cos\theta$

$x^2+y^2 = \frac{\sqrt{2}}{2}y - \frac{\sqrt{2}}{2}x$

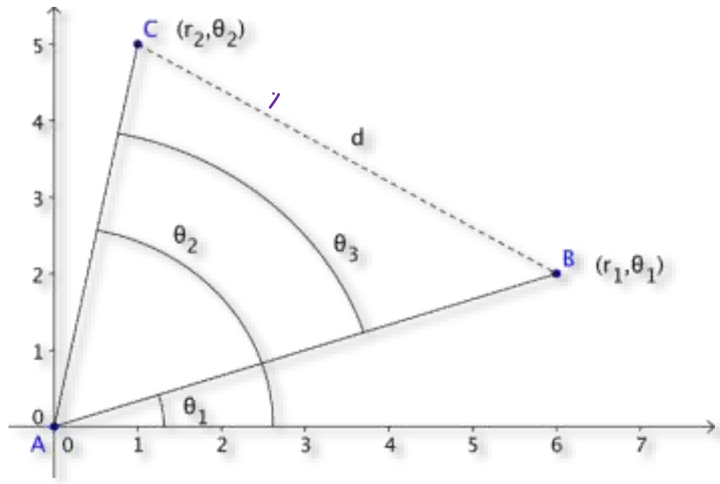
$x^2+\frac{\sqrt{2}}{2}x + y^2 - \frac{\sqrt{2}}{2}y = 0$

$x^2+\frac{\sqrt{2}}{2}x+\frac{1}{8} + y^2-\frac{\sqrt{2}}{2}y+\frac{1}{8} = \frac{1}{8}+\frac{1}{8}$

$\left(x+\frac{\sqrt{2}}{4}\right)^2 + \left(x-\frac{\sqrt{2}}{4}\right)^2 = \frac{1}{4}$

THE POLAR DISTANCE FORMULA:

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$$



10) Find the distance from $\left(2, \frac{\pi}{6}\right)$ to $\left(-3, \frac{\pi}{4}\right)$.

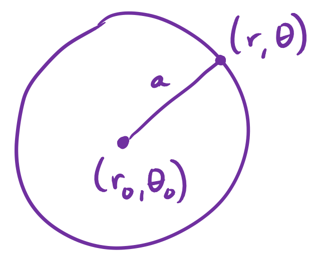
$d^2 = 2^2 + (-3)^2 - 2(2)(-3)\cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$

$d = \sqrt{4 + 9 + 12\cos\left(-\frac{\pi}{12}\right)}$

THE POLAR CIRCLE FORMULA:

$$a^2 = r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)$$

Radius a and center (r_0, θ_0)



Find the equation of a circle with:

11) radius 5 and center (0,0) origin

$\boxed{r = 5}$

12) radius 1 and center $\left(2, \frac{\pi}{6}\right)$

$1 = r^2 + 4 - 2(r)(2)\cos\left(\theta - \frac{\pi}{6}\right)$

$\boxed{1 = r^2 + 4 - 4r\cos\left(\theta - \frac{\pi}{6}\right)}$