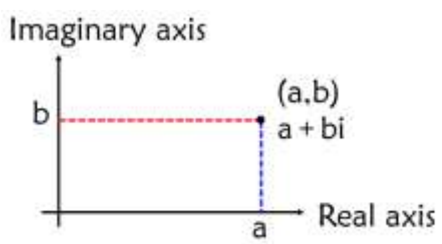
DEMOIVRE'S THEOREM

OBJECTIVES: 1) Write complex numbers in rectangular and polar form.
2) Multiply and divide complex numbers in polar form.

COMPLEX NUMBERS IN RECTANGULAR FORM

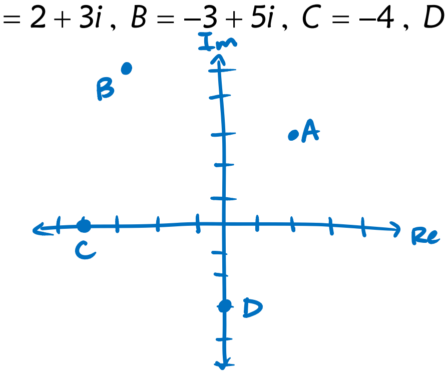
A complex number is any number that can be written in the form $a+bi$ where a and b are real numbers and i is the imaginary u nit. Therefore, every complex number $a+bi$ is associated with a unique ordered pair of real numbers (a,b) and vice versa.



COMPLEX PLANE

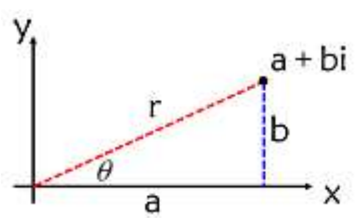
Graph the following numbers in the complex plane:

$A = 2+3i$, $B = -3+5i$, $C = -4$, $D = -3i$



COMPLEX NUMBERS IN POLAR FORM

Complex numbers can also be written in polar form, using polar-rectangular relationships.



$r = \sqrt{a^2+b^2}$ (r is the modulus) $\tan\theta = \frac{b}{a}$

$a = r\cos\theta$ $b = r\sin\theta$ θ is the argument

$a+bi = r\cos\theta + r\sin\theta\, i$

$= r(\cos\theta + i\sin\theta)$

r is positive, θ is not unique!

CIS FORM: If $z = a+bi$

$z = r \operatorname{cis}\theta$

(trig or polar form of complex number)

1) Express the complex number in rectangular form:

a) $z = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z = 3\operatorname{cis}\frac{\pi}{3}$ $r=3$ $\theta = \frac{\pi}{3}$

$a = r\cos\theta$ $b = r\sin\theta$

$a = 3\cos\frac{\pi}{3}$ $b = 3\sin\frac{\pi}{3}$

$a = 3\left(\frac{1}{2}\right)$ $b = 3\left(\frac{\sqrt{3}}{2}\right)$

rectangular form

$a+bi = \boxed{\frac{3}{2} + \frac{3\sqrt{3}}{2}i}$

b) $z = 6\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$a = 6\cos\frac{5\pi}{3}$ $b = 6\sin\frac{5\pi}{3}$

$a = 6\cdot\left(\frac{1}{2}\right)$ $b = 6\left(-\frac{\sqrt{3}}{2}\right)$

$a = 3$ $b = -3\sqrt{3}$

$a+bi = \boxed{3 - 3\sqrt{3}\,i}$

2) Express the complex number in trigonometric form:

a) $z = -\sqrt{2} + i\sqrt{2}$

$r = \sqrt{a^2+b^2}$ $\quad r = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2}$ $\quad r = \sqrt{4}$

$r = 2$

$\tan\theta = \frac{\sqrt{2}}{-\sqrt{2}} = -1$ $\quad \theta = \frac{3\pi}{4}$ or $\cancel{\frac{7\pi}{4}}$

$z = a + bi$ $\quad a = r\cos\theta$ $\quad b = r\sin\theta$

$z = 2\cos\frac{3\pi}{4} + 2\sin\frac{3\pi}{4} i$

$z = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ $\quad \boxed{z = 2 \operatorname{cis} \frac{3\pi}{4}}$

b) $z = -4 + 3i$

$r = \sqrt{4^2 + 3^2} = 5$ $\quad \tan\theta = \frac{3}{-4}$ $\quad \theta \approx -36.87$

$\theta \approx 143.1$

$\boxed{z = 5 \operatorname{cis} 143.1^\circ}$

MULTIPLYING COMPLEX NUMBERS

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Find $z_1 \bullet z_2$.

$$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 \cdot r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$$

$$= r_1 \cdot r_2(\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2} + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2))$$

$$= r_1 \cdot r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

If two complex numbers have polar form $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$z_1 \bullet z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$ (to multiply complex numbers, multiply the moduli and add the arguments)

$r_1 \cdot r_2 \operatorname{cis}(\theta_1 + \theta_2)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$ (to divide complex numbers, divide the moduli and subtract the arguments)

$\frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$

3) If $z = 8\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$ and $w = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ find $z \bullet w$ and $\frac{z}{w}$. Express each answer in trig form and in rectangular form.

$z \cdot w = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

$= 8 \cdot 4 \operatorname{cis}\left(\frac{5\pi}{3} + \frac{2\pi}{3}\right)$

$= \boxed{32 \operatorname{cis}\left(\frac{7\pi}{3}\right)}$ (Trig form)

$= 32\cos\frac{7\pi}{3} + 32i\sin\frac{7\pi}{3}$

$= 32 \cdot \frac{1}{2} + 32i \cdot \frac{\sqrt{3}}{2} = \boxed{16 + 16\sqrt{3}i}$ Rectangular form

$\frac{z}{w} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$

$\frac{z}{w} = \frac{8}{4} \operatorname{cis}\left(\frac{5\pi}{3} - \frac{2\pi}{3}\right)$

$\frac{z}{w} = \boxed{2 \operatorname{cis} \pi}$ Trig form

$= 2\cos\pi + i\sin\pi$

$= 2(-1) + i\,0 = \boxed{-2}$ rectangular form