DEMOIVRE'S THEOREM

OBJECTIVES: 1) Use DeMoivre's Theorem in binomial expansions involving complex numbers.
2) Find the nth roots of complex numbers.

DEMOIVRE'S THEOREM

Let $z = r(\cos\theta + i\sin\theta)$. Find z^2.

$z \cdot z = r(\cos\theta + i\sin\theta) \cdot r(\cos\theta + i\sin\theta)$

$= r \cdot r\,(\cos(\theta+\theta) + i\sin(\theta+\theta))$

$= r^2\,(\cos(2\theta) + i\sin(2\theta))$

$= r^2 \operatorname{cis} 2\theta$

DEMOIVRE'S THEOREM:
Let n be a natural number. Then

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$ $(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta)$

IN OTHER WORDS: To take the nth power of a complex number, take the nth root of the modulus and multiply the argument by n.

1) Find $\left(\sqrt{2} - i\sqrt{2}\right)^5$. Express your answer in rectangular form.

$r = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2$ $\tan\theta = \frac{-\sqrt{2}}{2}$ $\tan\theta = -1$ $\theta = \cancel{\frac{3\pi}{4}}, \frac{7\pi}{4}$

$(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}\theta n$

$\left(2\operatorname{cis}\frac{7\pi}{4}\right)^5 = 2^5 \operatorname{cis} 5\left(\frac{7\pi}{4}\right) = 32\operatorname{cis}\frac{35\pi}{4} = 32\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= 32\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \boxed{-16\sqrt{2} + 16\sqrt{2}i}$

2) Find $\left(\frac{1}{2} - \frac{1}{2}i\right)^{10}$.

$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$ $\tan\theta = \frac{-\frac{1}{2}}{\frac{1}{2}} = -1$ $\theta = \cancel{\frac{3\pi}{4}}, \frac{7\pi}{4}$

$\left(\frac{1}{2} - \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10} \operatorname{cis}\left(\frac{7\pi}{4} \cdot 10\right) = \frac{2^5}{2^{10}} \operatorname{cis}\left(\frac{35\pi}{2}\right) = \frac{1}{32}\operatorname{cis}\left(\frac{3\pi}{2}\right)$

$= \frac{1}{32}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 0 + \frac{1}{32} \cdot i \cdot -1 = \boxed{\frac{-1}{32}i}$

NTH ROOTS OF COMPLEX NUMBERS:

If n is a natural number and $z^n = w$, then z is an nth root of w.

We use the fact that expressing a complex number is not unique:

$$r(\cos\theta + i\sin\theta) = r(\cos(\theta + 2\pi n) + i\sin(\theta + 2\pi n))$$

3) Find all complex cube roots of 27i.

$z^3 = 27i$ $\quad r = \sqrt{0^2 + 27^2} = 27$ $\quad \tan\theta = \frac{27}{0}$ undefined $\theta = \frac{\pi}{2}, \cancel{\frac{3\pi}{2}}$

27i in polar form: $27i = 27\,\text{cis}\,\frac{\pi}{2}$

$(r\,\text{cis}\,\theta)^3 = 27\,\text{cis}\,\frac{\pi}{2}$

$r^3\,\text{cis}\,3\theta = 27\,\text{cis}\,\frac{\pi}{2}$

$r = 3$ $\quad 3\theta = \frac{\pi}{2} + 2\pi k$ $\quad \theta = \frac{\pi}{6} + \frac{2\pi}{3}k$

$z_1 = r\,\text{cis}\,\theta_1$ $\quad z_1 = 3\,\text{cis}\,\frac{\pi}{6} = \boxed{\frac{3\sqrt{3}}{2} + \frac{3}{2}i}$

$z_2 = r\,\text{cis}\,\theta_2$ $\quad z_2 = 3\,\text{cis}\,\frac{5\pi}{6} = \boxed{-\frac{3\sqrt{3}}{2} + \frac{3}{2}i}$

$z_3 = r\,\text{cis}\,\theta_3$ $\quad z_3 = 3\,\text{cis}\,\frac{3}{2}\pi = 0 + 3(-1)i$

$= \boxed{-3i}$

4) Find the fourth roots of -16.

$z^4 = -16$

$z^4 = 16\,\text{cis}\,\pi$

$r^4\,\text{cis}\,4\theta = 16\,\text{cis}\,\pi$

$r = 2$ $\quad 4\theta = \pi + 2\pi n$

$\theta = \frac{\pi}{4} + \frac{\pi}{2}n$

$z_1 = 2\,\text{cis}\,\frac{\pi}{4} = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \boxed{\sqrt{2} + i\sqrt{2}}$

$z_2 = 2\,\text{cis}\,\frac{3\pi}{4} = 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \boxed{-\sqrt{2} + i\sqrt{2}}$

$z_3 = 2\,\text{cis}\,\frac{5\pi}{4} = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \boxed{-\sqrt{2} - i\sqrt{2}}$

$z_4 = 2\,\text{cis}\,\frac{7\pi}{4} = 2\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \boxed{\sqrt{2} - i\sqrt{2}}$

5) Find the fourth roots of $2 + 2\sqrt{3}i$ $\quad z^4 = 2 + 2\sqrt{3}i$

$r = \sqrt{2^2 + (2\sqrt{3})^2} = 4$ $\quad \tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3}$ $\quad \theta = \frac{\pi}{3}, \cancel{\frac{4\pi}{3}}$

$2 + 2\sqrt{3}i = 4\,\text{cis}\,\frac{\pi}{3}$

$r^4\,\text{cis}\,4\theta = 4\,\text{cis}\,\frac{\pi}{3}$

$r = \sqrt{2}$ $\quad 4\theta = \frac{\pi}{3} + 2\pi k$ $\quad \theta = \frac{\pi}{12} + \frac{\pi}{2}k$

$z_1 = \sqrt{2}\,\text{cis}\,\frac{\pi}{12}$ $\quad z_4 = \sqrt{2}\,\text{cis}\,\frac{19}{12}\pi$

$z_2 = \sqrt{2}\,\text{cis}\,\frac{7\pi}{12}$

$z_3 = \sqrt{2}\,\text{cis}\,\frac{13\pi}{12}$