DETERMINANTS AND CRAMER'S RULE

OBJECTIVES: 1) Find the determinant of a 2 x 2 and 3 x 3 matrix.
2) Use Cramer's Rule to solve a system of equations in 2 or more variables.

DETERMINANTS

The way we take a determinant of a 2 × 2 matrixes is shown below:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Ex 1) $\begin{bmatrix} 6 & -3 \\ 9 & 3 \end{bmatrix}$ $\begin{vmatrix} 6 & -3 \\ 9 & 3 \end{vmatrix} = 18 - (-3 \cdot 9)$

$18 + 27$

$\boxed{45}$

The way to find the determinant of a 3 x 3 matrix is shown below:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Ex 2) $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 3 & -5 \\ 2 & -1 & 4 \end{bmatrix}$

$\det A = 3\begin{vmatrix} 3 & -5 \\ -1 & 4 \end{vmatrix} - (-2)\begin{vmatrix} 5 & -5 \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} 5 & 3 \\ 2 & -1 \end{vmatrix}$

$\det A = 3(12 - 5) + 2(20 + 10) + 1(-5 - 6)$

$\det A = 3(7) + 2(30) + 1(-11)$

$\boxed{\det A = 70}$

CRAMER'S RULE

For a two by two system, $\begin{cases} Ax + By = C \\ Dx + Ey = F \end{cases}$

$$x = \frac{\begin{vmatrix} C & B \\ F & E \end{vmatrix}}{\begin{vmatrix} A & B \\ D & E \end{vmatrix}} = \frac{CE - BF}{AE - BD} \quad \text{and} \quad y = \frac{\begin{vmatrix} A & C \\ D & F \end{vmatrix}}{\begin{vmatrix} A & B \\ D & E \end{vmatrix}} = \frac{AF - DC}{AE - BD}$$

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D}$$

Why does this work?

$E(Ax + By = C)$
$-B(Dx + Ey = F)$

$AEx + BEy = CE$
$-BDx - BEy = -BF$

$(AE - BD)x = CE - BF$

$x = \frac{CE - BF}{AE - BD}$

$-D(Ax + By = C)$
$A(Dx + Ey = F)$

$-ADx - BDy = -CD$
$ADx + AEy = AF$

$(AE - BD)y = AF - CD$

$y = \frac{AF - CD}{AE - BD}$

THREE VARIABLE SYSTEM

Ex 3) $\begin{cases} -3x + 2y - 6z = 6 \\ 5x + 7y - 5z = 6 \\ x + 4y - 2z = 8 \end{cases}$

$\text{Det}\begin{bmatrix} -3 & 2 & -6 \\ 5 & 7 & -5 \\ 1 & 4 & -2 \end{bmatrix}$

$x = \frac{D_x}{D} = \frac{172}{-86}$

$\text{Det}_x\begin{bmatrix} 6 & 2 & -6 \\ 6 & 7 & -5 \\ 8 & 4 & -2 \end{bmatrix}$

$x = -2$

$y = \frac{D_y}{D} = \frac{-258}{-86}$

$\text{Det}_y\begin{bmatrix} -3 & 6 & -6 \\ 5 & 6 & -5 \\ 1 & 8 & -2 \end{bmatrix}$

$y = 3$

$\text{Det}_z\begin{bmatrix} -3 & 2 & 6 \\ 5 & 7 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

$z = \frac{D_z}{D} = \frac{-86}{-86}$

$z = 1$