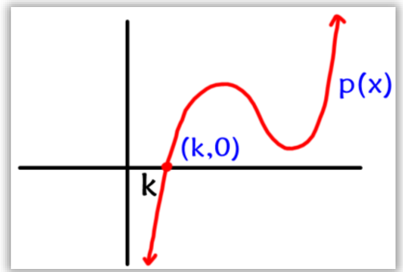
THE REMAINDER THEOREM AND FACTOR THEOREM

OBJECTIVES: 1) Check for a zero or a root by using the definition of a root.
2) Use remainder theorem to check for a factor.
3) Use factor theorem to solve polynomial equations.

DEFINITION

Given a polynomial $p(x)$, a number "k" is a **zero** (**root**) of $p(x)$ if $p(k) = 0$.



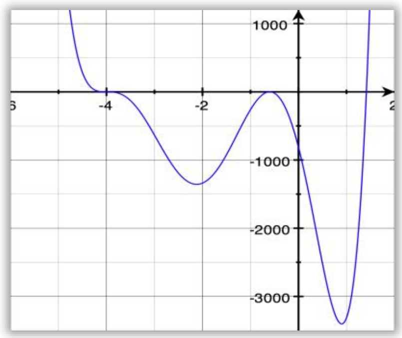
1) Is $\sqrt{3}$ a root of $f(x) = 2x^4 + x^2 - 21$

$f(\sqrt{3}) = 2(\sqrt{3})^4 + (\sqrt{3})^2 - 21$
$= 2(3^2) + 3 - 21$
$= 2(9) + 3 - 21$
$= 0$

$\boxed{\text{Yes! } f(\sqrt{3}) = 0!}$

MULTIPLICTY OF ROOTS

Let $f(x) = \left(x - \sqrt{2}\right)(5x+3)^2(x+4)^3$.



$(x - \sqrt{2})$
$\sqrt{2} \rightarrow$ single root

$(5x+3)^2$
$-\frac{3}{5} \rightarrow$ double root

$(x+4)^3$
$-4 \rightarrow$ "root of multiplicity 3"

FACTOR THEOREM: $(x-b)$ is a factor of $f(x)$ if and only if $f(b) = 0$.

In other words: a) If $(x-r)$ is a factor of $f(x)$, then $f(r) = 0$

b) If $f(r) = 0$, then $(x-r)$ is a factor of $f(x)$.

REMAINDER THEOREM: When a polynomial $f(x)$ is divided by $(x-r)$, then the remainder is $f(r)$.

$f(r)!!$

2) Is 2 a root of $x^3 - 4x^2 + 3x + 7$?

$f(2) = 2^3 - 4(2)^2 + 3(2) + 7$
$= 8 - 16 + 6 + 7$
$= 5$

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 3 & 7 \\ & & 2 & -4 & -2 \\ \hline & 1 & -2 & -1 & \boxed{5} \end{array}$$

Synthetic substitution

$f(2) = 5 \Rightarrow 2$ is not a root!

3) Find $f(-3)$ to determine whether $(x+3)$ is a factor of $f(x) = 2x^4 + 5x^3 - 2x - 8$.

$f(-3) = 2(-3)^4 + 5(-3)^3 - 2(-3) - 8 = 25$

$f(-3) = 25$

$$\begin{array}{r|rrrrr} -3 & 2 & 5 & 0 & -2 & -8 \\ & & -6 & 3 & -9 & 33 \\ \hline & 2 & -1 & 3 & -11 & \boxed{25} \end{array}$$

$\boxed{(x+3) \text{ is not a factor!}}$

4) Solve $f(x) = x^4 - 2x^3 - 10x^2 + 4x + 16 = 0$ knowing –2 and √2 are roots.

$(x+2)(x-\sqrt{2})$ are factors!

$$\begin{array}{r|rrrrr} -2 & 1 & -2 & -10 & 4 & 16 \\ & & -2 & 8 & 4 & -16 \\ \hline & 1 & -4 & -2 & 8 & \boxed{0} \end{array}$$

$x^3 - 4x^2 - 2x + 8$

$$\begin{array}{r|rrrr} \sqrt{2} & 1 & -4 & -2 & 8 \\ & & \sqrt{2} & -4\sqrt{2}+2 & -8 \\ \hline & 1 & -4+\sqrt{2} & -4\sqrt{2} & \boxed{0} \end{array}$$

$x^2 + (-4+\sqrt{2})x - 4\sqrt{2} = 0$

$x^2 - 4x + \sqrt{2}x - 4\sqrt{2} = 0$

$(x+\sqrt{2})(x-4) = 0$

$x = -\sqrt{2}$ $\quad x = 4$ $\quad$ and original roots!
$x = -2$ $\quad x = \sqrt{2}$

5) Find **a** polynomial with degree 3 and a root of 1 with multiplicity 2 and a root of –2.

$\boxed{f(x) = (x-1)^2(x+2)}$