RATIONAL AND IRRATIONAL ROOTS

OBJECTIVES: 1) Find all roots of a polynomial equation.
2) Simplify powers of *i*.

THE RATIONAL ROOTS THEOREM:

For every polynomial of the form: $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$

first find all the factors of a_0 (call them p)

and all the factors of a_n (called q)

then a *list* of possible **rational** roots is found by creating all possible p/q 's.

THE UPPER BOUND AND LOWER BOUND THEOREM

If b is positive and b

$b\,\big|\;$ + + + + + +

then b is an upper bound;

there are no rational roots above b.

If b is negative and b

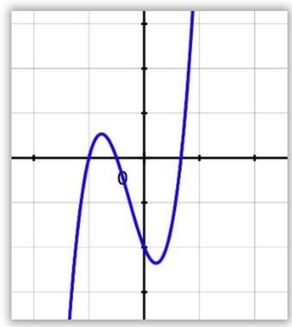
$b\,\big|\;$ + − + − + − OR $b\,\big|\;$ − + − + − +

then b is a lower bound;

there are no rational roots below b.

1) Find all roots of $6x^3 + 5x^2 - 3x - 2 = 0$.

$\frac{p}{q} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm 2, \pm \frac{2}{3}$ } List of ALL possible rational roots



$$\begin{array}{r|rrrr} -1 & 6 & 5 & -3 & -2 \\ & & -6 & 1 & 2 \\ \hline & 6 & -1 & -2 & \boxed{0} \end{array}$$

$6x^2 - x - 2 = 0$

$x = \frac{1 \pm \sqrt{1-4(6)(-2)}}{2(6)}$

$x = \frac{1 \pm \sqrt{49}}{12}$

$x = -\frac{1}{2}, \frac{2}{3}$

All roots:
$x = -1, -\frac{1}{2}, \frac{2}{3}$

2) Show that no rational roots exist for the equation $\frac{1}{4}x^4 - \frac{3}{4}x^3 + \frac{17}{4}x^2 + 4x + 5 = 0$.

$x^4 - 3x^3 + 17x^2 + 16x + 20 = 0$

$\frac{p}{q} = \frac{\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20}{\pm 1} = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

$$\begin{array}{r|rrrrr} 1 & 1 & -3 & 17 & 16 & 20 \\ & & 1 & -2 & 15 & 31 \\ \hline & 1 & -2 & 15 & 31 & \boxed{51} \end{array}$$

$$\begin{array}{r|rrrrr} 2 & 1 & -3 & 17 & 16 & 20 \\ & & 2 & -2 & 30 & 92 \\ \hline & 1 & -1 & 15 & 46 & \boxed{112} \end{array}$$

$$\begin{array}{r|rrrrr} 4 & 1 & -3 & 17 & 16 & 20 \\ & & 4 & 4 & 84 & 400 \\ \hline & 1 & 1 & 21 & 100 & \boxed{420} \\ & + & + & + & + & + \end{array}$$

∴ 4 is an upper bound

$$\begin{array}{r|rrrrr} -1 & 1 & -3 & 17 & 16 & 20 \\ & & -1 & 4 & -21 & 5 \\ \hline & 1 & -4 & 21 & -5 & \boxed{25} \\ & + & - & + & - & + \end{array}$$

∴ −1 is a lower bound

None of the possible rational roots > 4 can be roots.

None of the possible roots < −1 can be roots. Zero is also not a root of the equation, ∴ there are

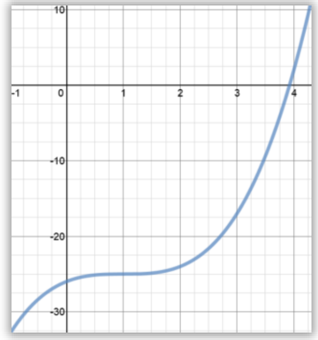
NO rational roots.

THE LOCATION THEOREM:

For every polynomial of the form: $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$

If a and b are real numbers such that f(a) and f(b) have opposite signs, then the equation f(x)=0 has at least one real root between a and b.

3) Find the roots between successive hundredths using $x^3 - 3x^2 + 3x - 26 = 0$.



$f(3) = -17$
$f(4) = +2$ > sign change! ∴ a root exists between 3 & 4

$f(3.8) = -3.048$
$f(3.9) = -.611$
$f(4.0) = 2$ > root between 3.9 & 4.0

$f(3.91) = -.358$
$f(3.92) = -.103$
$f(3.93) = .154$ > sign change!

Root occurs between open interval (3.92, 3.93)

DIVIDING COMPLEX NUMBERS:

4) $\dfrac{7-8i}{2i} \cdot \dfrac{i}{i} = \dfrac{7i - 8i^2}{2i^2} = \boxed{\dfrac{7i+8}{-2}}$

5) $\dfrac{2-4i}{1+3i} \cdot \dfrac{1-3i}{1-3i}$

$\dfrac{2-6i-4i+12i^2}{1-9i^2} = \dfrac{-10i-10}{10} = \boxed{-i-1}$

6) $\dfrac{3+4i}{7-i} \cdot \dfrac{7+i}{7+i}$

$\dfrac{21+3i+28i+4i^2}{49-i^2} = \boxed{\dfrac{17+31i}{50}}$

SIMPLIFYING POWERS OF *i*

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = (i^2)i = -1i = -i$

$i^4 = (i^2)^2 = (-1)^2 = \boxed{1}$

7) $i^{24} = (i^2)^{12} = (-1)^{12} = \boxed{1}$

8) $i^{23} = (i^2)^{11} i = (-1)^{11} i = \boxed{-i}$

9) $i^{100} = (i^2)^{50} = (-1)^{50} = \boxed{1}$

10) $i^{49} = (i^2)^{24} i$
$(-1)^{24} i$
$\boxed{i}$

11) $i^{38} = (i^2)^{19} = (-1)^{19} = \boxed{-1}$

12) $\left(\sqrt[8]{5} \bullet i\right)^{24} =$
$(5^{\frac{1}{8}})^{24} i^{24}$
$5^3 i^{24}$
$\boxed{125}$

13) $\left(\sqrt[6]{3} \bullet i\right)^{18} =$
$(3^{\frac{1}{6}})^{18} \cdot i^{18}$
$3^3 \cdot (i^2)^9$
$27 \cdot -1 = \boxed{-27}$