MATHEMATICAL INDUCTION

OBJECTIVES: 1) Use mathematical induction to write proofs.

PRINCIPLE OF MATHEMATICAL INDUCTION

To illustrate how induction works, imagine that you are climbing an infinitely high ladder. How do you know whether you will be able to reach an arbitrarily high rung? Suppose you make the following two assertions about your climbing abilities:

1) I can definitely reach the first rung.

2) Once I get to any rung, I can always climb to the next one up.

If both statements are true, then by statement 1 you can get to the first one, and by statement 2, you can get to the second. By statement 2 again, you can get to the third, and fourth, etc. Therefore, you can climb as high as you wish. Notice that both of these assertions are necessary for you to get anywhere on the ladder. If only statement 1 is true, you have no guarantee of getting beyond the first rung. If only statement 2 is true, you may never be able to get started.

WRITING A PROOF BY INDUCTION

A proof by mathematical induction that a proposition P(n) is true for every positive integer n consists of two steps:

BASE CASE: Show that the proposition P(1) is true.

INDUCTIVE STEP: Assume that P(k) is true for an arbitrarily chosen positive integer k, and show that under that assumption, P(k+1) must be true.

From these two steps we conclude (by the principle of mathematical induction) that for all positive integers n, P(n) is true.

EXAMPLES

1) Use mathematical induction to show that the statement is true for all natural numbers:

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

Step 1) Show that P(1) is true.

$P(1)$: $1 = \frac{1(1+1)}{2} = \frac{1\cdot 2}{2} = 1$

$1=1$ ✓

Step 2) Show that P(k+1) is true.

Assume that P(k) is true: $1+2+\ldots+k = \frac{k(k+1)}{2}$

Show $P(k+1)$: $1+2+3+\ldots+k+k+1 = \frac{(k+1)[(k+1)+1]}{2}$ is true.

$$1+2+3+\ldots+k+k+1 = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k^2+k}{2} + \frac{2k+2}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)[(k+1)+1]}{2} \checkmark$$

$\therefore$ $P(n)$ is true for all natural numbers!

2) Use mathematical induction to show that the statement is true for all natural numbers:

$$1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Step 1) Show that $P(1)$ is true.

$P(1)$: $1 = \frac{1(2-1)(2+1)}{3} = \frac{1(1)(3)}{3} = 1 \checkmark$

Step 2) Show that $P(k+1)$ is true.

Assume that $P(k)$ is true: $1^2+3^2+5^2+\ldots+(2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$ is true.

We want to show that $P(k+1)$ is true:

$$1^2+3^2+5^2+\ldots+(2k-1)^2+(2(k+1)-1)^2 = \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

$$1^2+3^2+5^2+\ldots+(2k-1)^2+(2(k+1)-1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2(k+1)-1)^2$$

$$= \frac{(2k^2-k)(2k+1)}{3} + (2k+1)^2$$

factor:

$$\begin{array}{r|rrrr} -1 & 4 & 12 & 11 & 3 \\ & & -4 & -8 & -3 \\ \hline & 4 & 8 & 3 & \boxed{0} \end{array}$$

$4k^2+8k+3$

$(2k+1)(2k+3)$

$$= \frac{4k^3+2k^2-2k^2-k}{3} + \frac{3(4k^2+4k+1)}{3}$$

$$= \frac{4k^3+12k^2+11k+3}{3} \longrightarrow \text{factor: } \begin{array}{r|rrrr} -1 & 4 & 12 & 11 & 3 \\ & & -4 & -8 & -3 \\ \hline & 4 & 8 & 3 & \boxed{0} \end{array}$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3} = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3} \checkmark$$

Since $P(k+1)$ is true, $P(n)$ is true for all natural numbers.