(PART 1) 7.6 NOTES – FACTOR THEOREM

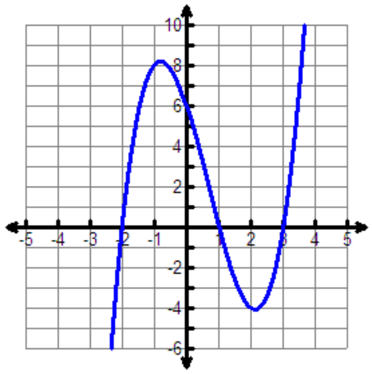
OBJECTIVES:

1) Identify the possible rational zeros of a polynomial function.
2) Find all the roots of a polynomial function.

ROOT: The roots of a polynomial are the points where the graph of the polynomial cross the x-axis. Roots can also be called zeros, solutions or intercepts, but they all mean the same thing!

Example: $f(x) = x^3 - 2x^2 - 5x + 6$

Zeros: –2, 1, 3



REVIEW:

1) Factor completely: $2x^2 + 8x + 6$.

$2(x^2+4x+3)$

$\boxed{2(x+1)(x+3)}$

2) Solve by factoring, $x^2 - x - 12 = 0$

$(x-4)(x+3)=0$

$\boxed{x=4,\ -3}$

3) Relationship between zeros and factors. Complete the chart.

Zeros	2	–3	-6	5	$-\sqrt{7}$
Factors	$(x-2)$	$(x+3)$	$(x+6)$	$(x-5)$	$(x+\sqrt{7})$

4) Consider the polynomial $f(x) = 4x^2 - 3x + 6$.

a. Divide the polynomial by $(x-2)$.

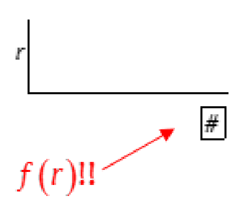
$$\begin{array}{r|rrr} 2 & 4 & -3 & 6 \\ & & 8 & 10 \\ \hline & 4 & 5 & 16 \end{array}$$

b) Evaluate $f(2)$. $f(2) = 4(2)^2 - 3(2) + 6 = 16 - 6 + 6 = 16$

c) How is the value of $f(2)$ related to the remainder we got in part a) ?

They are the same!

REMAINDER THEOREM: When a polynomial $f(x)$ is divided by $(x-r)$, then the remainder is $f(r)$.



FACTOR THEOREM: $(x-b)$ is a factor of $f(x)$ if and only if $f(b)=0$.

In other words: a) If $(x-r)$ is a factor of $f(x)$, then $\underline{f(r)=0}$

b) If $f(r)=0$, then $\underline{(x-r)}$ is a factor of $f(x)$.

$$\begin{array}{r|l} r & \\ \hline & \boxed{0} \end{array}$$

then $(x-r)$ factors $f(x)$

5) Is 2 a root of x^3-4x^2+3x+7?

$f(2) = 2^3 - 4(2)^2 + 3(2) + 7 = 8 - 16 + 6 + 7 = 5$ No, 2 is not a root.

6) Find $f(-3)$ to determine whether $(x+3)$ is a factor of $f(x)=2x^3+11x^2+18x+9$.

$f(-3) = 2(-3)^3 + 11(-3)^2 + 18(-3) + 9$

$= -54 + 99 - 54 + 9 = 0$

Yes, $(x+3)$ is a factor!

7) Is $(x-1)$ a factor of $f(x)=x^3-2x+1$?

$f(1) = 1 - 2(1) + 1 = 0$ Yes!

LISTING ALL POSSIBLE RATIONAL ROOTS:

So let's take a look at the function, $f(x)=x^3-2x^2-5x+6$, our original example from before. From the graph, we already know what our factors and zeros should be, but we will now find them algebraically.

Example 1) $f(x)=x^3-2x^2-5x+6$

a) Look for factors of $\pm\frac{p}{q}$ where $\frac{p}{q}=\frac{\text{factors of the constant}}{\text{factors of the lead coefficient}}$

$\pm\frac{p}{q} = \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1} = \boxed{\pm 1, 2, 3, 6}$

$f(1) = 0$

$f(-1) = 8$

$f(2) = -4$

$f(-2) = 0$

$f(3) = 0$

$f(-3) = -24$

$f(6) = 120$

$f(-6) = -252$

FACTORING THE POLYNOMIAL:

b) Using the factor found from part a, use long/synthetic division to find other factors.

$f(1) = 0$ so $(x-1)$ is a factor of x^3-2x^2-5x+6

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

Factor: $x^2 - x - 6$

$(x-3)(x+2)$

$\boxed{(x-1)(x-3)(x+2)}$

notice: $(-1)(-3)(2) = 6$

$x \cdot x \cdot x = x^3$

Example 2) Factor: $f(x) = 4x^3 - 5x^2 - 23x + 6$ if $f(-2) = 0$.

a) List all of the possible rational zeros.
b) Factor the polynomial.

a) $\pm \frac{p}{q} = \frac{\pm 1, 2, 3, 6}{\pm 1, 2, 4} = \boxed{\pm 1, \frac{1}{2}, \frac{1}{4}, 2, 3, \frac{3}{2}, \frac{3}{4}, 6}$

b) $f(-2) = 0$ so $(x+2)$ is a factor

$$\begin{array}{r|rrrr} -2 & 4 & -5 & -23 & 6 \\ & & -8 & 26 & -6 \\ \hline & 4 & -13 & 3 & \boxed{0} \end{array}$$

Factor:
$4x^2 - 13x + 3$

$4x^2 - 12x - x + 3$

$4x(x-3) - 1(x-3)$

$(4x-1)(x-3)$

$\boxed{(x+2)(x-3)(4x-1)}$

Example 3) Factor: $f(x) = 4x^4 - 5x^3 - 26x^2 + 9x + 18$

a) List all of the possible rational zeros.
b) Factor the polynomial.

a) $\pm \frac{p}{q} = \frac{\pm 1, 2, 3, 6, 9, 18}{\pm 1, 2, 4} = \pm 1, \frac{1}{2}, \frac{1}{4}, 2, 3, \frac{3}{2}, \frac{3}{4}, 6, 9, \frac{9}{2}, \frac{9}{4}, 18$

$f(1) = 0$ so $(x-1)$ is a factor.

$$\begin{array}{r|rrrrr} 1 & 4 & -5 & -26 & 9 & 18 \\ & & 4 & -1 & -27 & -18 \\ \hline & 4 & -1 & -27 & -18 & \boxed{0} \end{array}$$

$$\begin{array}{r|rrrr} -2 & 4 & -1 & -27 & -18 \\ & & -8 & 18 & 18 \\ \hline & 4 & -9 & -9 & \boxed{0} \end{array}$$

$h(x) = 4x^4 - x^3 - 27x^2 - 18x$

$h(-2) = 0$ so $(x+2)$ is a factor

$4x^2 - 9x - 9$

$4x^2 - 12x + 3x - 9$

$4x(x-3) + 3(x-3) = (4x+3)(x-3)$

Answer:
$\boxed{(x-1)(x+2)(4x+3)(x-3)}$

Example 4) Factor: $f(x) = x^3 - 3x^2 - 16x - 12$ if $f(6) = 0$.

a) List all of the possible rational zeros.
b) Factor the polynomial.

a) $\pm \frac{p}{q} = \frac{\pm 1, 2, 3, 4, 6, 12}{\pm 1} = \pm 1, 2, 3, 4, 6, 12$

b) $f(6) = 0 \Rightarrow (x-6)$ is a factor

$$\begin{array}{r|rrrr} 6 & 1 & -3 & -16 & -12 \\ & & 6 & 18 & 12 \\ \hline & 1 & 3 & 2 & \boxed{0} \end{array}$$

$x^2 + 3x + 2$

$(x+1)(x+2)$

Answer:
$\boxed{(x-6)(x+1)(x+2)}$